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The Green's function for a linear energy chirped seeded free electron laser

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Abstract
In a free electron laser (FEL), the electron beam at the undulator entrance can have a correlated energy spread. In this paper, we derive an expression of the seeded FEL Green's function for the case of an electron beam having a linear energy chirp, within the one-dimensional Vlasov–Maxwell model. This Green's function allows to evaluate the FEL electromagnetic radiation in both the lethargy and the exponential growth regime, without the asymptotic approximation introduced in previous works. We show a comparison between the proposed expression for the Green's function and the one obtained with a saddle point approximation, for both cases of short and long undulators.

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1. Introduction

For an x-ray free electron laser (FEL), a high-quality electron bunch with low emittance, high peak current and energy is needed [1]. During the phases of acceleration, bunch compression and transportation, the electron beam is subject to radio frequency curvature and to wakefields effects. Thus, the energy profile of the electron beam can undergo modifications, and in particular it can experience a linear energy chirp, which has important electromagnetic effects on the FEL process.

In this paper, we derive the Green's function for an electron beam having initial linear energy chirp, by solving the set of 1D Vlasov–Maxwell equations, which describe the motion of the electrons and the evolution of the radiation field. The effects of the initial linear energy chirp on the FEL performance, and possible short-pulses generation, have been studied for the self-amplified spontaneous emission (SASE) FEL [2, 3] and for a seeded FEL as well [4]. However, in [2–4], the Green's function is derived by using a saddle point approximation. This kind of solution describes the FEL electromagnetic radiation in the exponential growth regime with good accuracy, but is not satisfactory in the lethargy regime. The formula that we are proposing allows us to evaluate with better accuracy the behavior of the Green's function in the exponential growth regime, thus giving the correct characterization of interesting properties of the FEL light such as frequency shift and frequency chirp. Moreover the FEL radiation can be correctly evaluated also in the lethargy regime, thus giving a good characterization of the FEL pulse also for short undulators.

The paper is organized as follows. In section 2, the coupled Vlasov–Maxwell equations are introduced and solved: this gives an integral representation for the FEL Green's function. Subsequently, an exact series expansion for the Green's function is obtained by performing an inverse Laplace transform in analytical form. In section 3, two test cases are considered, involving a short and a long undulator, respectively, and the results obtained with the new formula for the Green's function are compared to those obtained with the asymptotic approximated expressions. Section 4 concludes the work. Mathematical details are given in the appendix.

2. Solution of the Vlasov–Maxwell equations with an initial value problem

In order to study the start-up of a seeded FEL amplifier, we analyze the set of the Vlasov and Maxwell equations, which describe the interaction between the relativistic electrons of a beam and the electromagnetic field [6], assuming the presence of an initial linear energy chirp. We solve this set of equations providing an exact series expansion for the Green's function.

2.1. Coupled Vlasov–Maxwell equations

Throughout this paper, we adopt the notations of [2, 4–6]. In particular, we use the dimensionless variables $Z = k_w z$ and $\theta = (k_0 + k_w)z - \omega_0 t$, where z is the longitudinal coordinate, $k_w = 2\pi/\lambda_w$, with λ_w the undulator period, $k_0 = 2\pi/\lambda_0$, with λ_0 the radiation wavelength, and $\omega_0 = k_0 c$, with c the velocity of light in the vacuum. As a measure of the energy deviation, we also introduce the quantity $p = 2(\gamma - \gamma_0)/\gamma_0$, where γ is the Lorentz factor of an electron of the beam and γ_0 the Lorentz factor in resonance condition. For a planar undulator, the latter quantity satisfies the relation

$$\lambda_0 \approx \lambda_w \frac{1 + \frac{K^2}{2}}{2\gamma_0^2}, \tag{1}$$

where the undulator parameter is $K \approx 93.4 B_w \lambda_w$, with B_w the peak magnetic field in tesla and λ_w the undulator period in meters. The electron distribution function is denoted as $\psi(\theta, p, Z)$ and the FEL electric field is written in the form $E(\theta, Z) = A(\theta, Z)\,\mathrm{e}^{\mathrm{i}(\theta - Z)}$, with $A(\theta, Z)$ being the slow varying envelope function. Following [4], the one dimensional linearized Vlasov–Maxwell equations are expressed by

$$\frac{\partial \psi(\theta, p, Z)}{\partial Z} + p \frac{\partial \psi(\theta, p, Z)}{\partial \theta} - \frac{2D_2}{\gamma_0^2}\left(A(\theta, Z)\,\mathrm{e}^{\mathrm{i}\theta} + A^*(\theta, Z)\,\mathrm{e}^{-\mathrm{i}\theta}\right)\frac{\partial \psi_0(\theta, p, Z)}{\partial p} = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial \theta}\right) A(\theta, Z) = \frac{D_1}{\gamma_0}\,\mathrm{e}^{-\mathrm{i}\theta} \int \mathrm{d}p\, \psi(\theta, p, Z), \tag{3}$$

where the asterisk denotes the complex conjugate, the integral is defined on the whole p domain, and, in SI units:

$$D_1 = \frac{e a_w n_0 [JJ]}{2\sqrt{2} k_w \varepsilon_0} \qquad \text{and} \qquad D_2 = \frac{e a_w [JJ]}{\sqrt{2} k_w m c^2}, \tag{4}$$

with e and m being the charge and the mass of the electron, respectively, ε_0 the vacuum permittivity, n_0 the electron beam density and $[JJ] = J_0\left[a_w^2/2\left(1+a_w^2\right)\right] - J_1\left[a_w^2/2\left(1+a_w^2\right)\right]$, where $a_w = K/\sqrt{2}$ is the dimensionless rms undulator parameter, while J_0 and J_1 are the Bessel functions of the first kind of orders 0 and 1, respectively. Finally, the function ψ_0 in (2) is defined as a solution of the equation $\frac{\partial\psi}{\partial Z} + p\frac{\partial\psi}{\partial\theta} = 0$.

We assume that the electron beam has a linear chirp, described by the relation

$$\gamma = \gamma_0 + \left.\frac{\partial\gamma}{\partial t}\right|_{\substack{t=0\\z=0}} t \tag{5}$$

and solve the system of equations (2) and (3) assuming

$$\psi_0 = \delta(p + \mu\theta_0), \tag{6}$$

where $\theta_0 = \theta - pZ$, $\mu = \frac{2}{\gamma_0\omega_0}\frac{\partial\gamma}{\partial t}\Big|_{\substack{t=0\\z=0}}$ and setting $\psi = \delta(p + \mu\theta_0) + \psi_1$, with ψ_1 a small perturbation of ψ_0.

Solving the Vlasov equation (2), substituting the solution into the Maxwell equation (3), assuming $\mu Z \ll 1$ and taking into account the arrival times of the single electrons, we have [4]

$$\left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial\theta}\right) A(\theta, Z) = \frac{D_1}{\gamma_0}\sum_j \mathrm{e}^{-\mathrm{i}\theta_j + \mathrm{i}\mu\theta_j Z}\delta(\theta - \theta_j)$$
$$+\,\mathrm{i}(2\rho)^3 \int_0^Z \mathrm{d}Z'(Z - Z')\,\mathrm{e}^{\mathrm{i}\mu\theta(Z-Z')}A(\theta, Z'), \tag{7}$$

where $\theta_j = -\omega_0 t_j$ and $(2\rho)^3 = 2D_1D_2/\gamma_0^3$ with ρ being the Pierce parameter [7].

Introducing the Laplace transform

$$f(\theta, s) = \int_0^\infty \mathrm{d}Z\,\mathrm{e}^{-sZ}A(\theta, Z). \tag{8}$$

Equation (7) is then rewritten in the Laplace complex frequency domain as

$$\frac{\partial f(\theta, s)}{\partial\theta} + \left(s - \frac{\mathrm{i}(2\rho)^3}{(s - \mathrm{i}\mu\theta)^2}\right) f(\theta, s) = A(\theta, 0) + \frac{D_1}{\gamma_0}\sum_j \frac{\mathrm{e}^{-\mathrm{i}\theta_j}\delta(\theta - \theta_j)}{s - \mathrm{i}\mu\theta_j}, \tag{9}$$

which yields the solution

$$f(\theta, s) = \int_{-\infty}^{\theta} \mathrm{d}\theta'\,\mathrm{e}^{-s(\theta-\theta') + \int_{\theta'}^{\theta}\frac{\mathrm{i}(2\rho)^3}{(s-\mathrm{i}\mu\theta'')^2}\mathrm{d}\theta''} \times \left[A(\theta', 0) + \frac{D_1}{\gamma_0}\sum_j \frac{\mathrm{e}^{-\mathrm{i}\theta_j}\delta(\theta' - \theta_j)}{s - \mathrm{i}(\mu\theta_j)}\right]. \tag{10}$$

The term $A(\theta', 0)$ in equation (10) characterizes the initial seed for a seeded FEL, while the second term in the square bracket models the shot noise source for the SASE FEL. Since we are considering only the seeded FEL, in the following we will keep only the term depending on the initial seed. Thus, equation (10) becomes

$$f(\theta, s) = \int_{-\infty}^{\theta} \mathrm{d}\theta'\,\mathrm{e}^{-s(\theta-\theta') + \frac{\mathrm{i}(2\rho)^3(\theta-\theta')}{(s-\mathrm{i}\mu\theta)(s-\mathrm{i}\mu\theta')}}A(\theta', 0). \tag{11}$$

The inverse Laplace transform gives the FEL field envelope along the undulator, that is

$$A(\theta, Z) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{d}s\,\mathrm{e}^{sZ}\int_{-\infty}^{\theta} \mathrm{d}\theta'\,\mathrm{e}^{-s(\theta-\theta') + \frac{\mathrm{i}(2\rho)^3(\theta-\theta')}{(s-\mathrm{i}\mu\theta)(s-\mathrm{i}\mu\theta')}}A(\theta', 0). \tag{12}$$

Introducing the variable $\xi = \theta - \theta'$ and manipulating, equation (12) can be written as

$$A(\theta, Z) = \rho\int_0^\infty \mathrm{d}\xi\,A(\theta - \xi, 0)\,\mathcal{G}(\theta, \xi, Z, \mu)\,, \tag{13}$$

where the Green's function $\mathcal{G}(\theta, \xi, Z, \mu)$ is defined as

$$\mathcal{G}(\theta, \xi, Z, \mu) = \frac{1}{\rho} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi \mathrm{i}} \, \mathrm{e}^{s(Z-\xi) + \frac{\mathrm{i}(2\rho)^3 \xi}{(s - \mathrm{i}\mu\theta)(s - \mathrm{i}\mu(\theta - \xi))}}. \tag{14}$$

For more convenience, the Green's function in equation (14) is rewritten using the notations adopted in [3, 5]: $\hat{z} = 2\rho Z$, $\hat{s} = \rho\theta$, $\hat{\xi} = \rho\xi$, $\hat{\alpha} = -\mu/(2\rho^2)$, and $\hat{p} = s/(2\rho)$. This yields

$$\mathcal{G}\left(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}\right) = \frac{1}{\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \mathrm{d}\hat{p} \, \mathrm{e}^{\hat{p}(\hat{z} - 2\hat{\xi}) + \frac{2\mathrm{i}\hat{\xi}}{(\hat{p} + \mathrm{i}\hat{\alpha}\hat{s})(\hat{p} + \mathrm{i}\hat{\alpha}(\hat{s} - \hat{\xi}))}}, \tag{15}$$

where σ belongs to the convergence region.

2.2. *Series expansion of the Green's function*

An approximation of the Green's function in equation (15) was found for a SASE FEL process in the case of a linear energy chirped electron beam [2] and for a seeded FEL process with both linear chirp and energy curvature [5], using a saddle point approximation method. Here we present an exact expression for the integral in (15), obtained by exploiting the residual theorem in conjunction with a series expansion of the integrand function. At first, we introduce an auxillary integral that fullfills the conditions of the Jordan's Lemma and is related to the original integral. Subsequently, since the integrand function presents two essential singularities, we evaluate the two residuals values by expanding it in Laurent series in the neighborhood of each singularity.

Following the procedure described in the appendix, the Green's function is expressed by the following series expression:

$$\mathcal{G}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \sum_{t=0}^{+\infty} \hat{\alpha}^t \sum_{h=0}^{+\infty} \frac{\mathrm{i}^{1+h+t} 2^{2+h} (\hat{z} - 2\hat{\xi})^{1+2h+t} \sum_{w=0}^{t} \frac{(-1)^w \hat{s}^w \hat{\xi}^{1+h+t-w} (h+t-w)!}{w!(t-w)!(1+2h+t-w)!}}{h!(1+h)!} + \delta(\hat{\xi} - \hat{z}/2), \tag{16}$$

where δ is the Dirac delta function.

In order to validate the expression in equation (16), we performed the inverse Laplace transform in equation (15) numerically. This can be done choosing, in the complex plane, a circular path including the singularities, to evaluate the line integral. The algorithm to perform the integration has to take into account some numerical issues: in fact, the integration path in the complex plane should not be too close to the singularities $\hat{p}_1 = -\mathrm{i}\hat{\alpha}\hat{s}$ and $\hat{p}_2 = \mathrm{i}\hat{\alpha}(\hat{\xi} - \hat{s})$, and furthermore the term $\mathrm{e}^{\hat{p}(\hat{z} - 2\hat{\xi})}$ should not be too large compared to the value of the integral. Since the evaluation of the Green's function for a single value of $\hat{s}$ requires the evaluation of the line integral for different $\hat{\xi}$ from 0 to $\hat{z}/2$, the integration path should be chosen carefully for each value of $\hat{\xi}$, in order to have a reliable result. Furthermore, the numerical integration does not allow us to evaluate the solution when $\hat{\xi}$ is close to $\hat{z}/2$, because it cannot represent a Dirac delta. This comparison not only proved that the expression given in equation (16) is correct but also showed the advantage of having a series expansion instead of using a numerical integration.

3. Comparison between the proposed series expression for the Green's function and the saddle point approximation

In this section, we give some amplitude and phase plots of the Green's function, calculated by using equation (16). Such results are compared with those obtained in [5] via saddle point approximation, setting the energy curvature to zero.

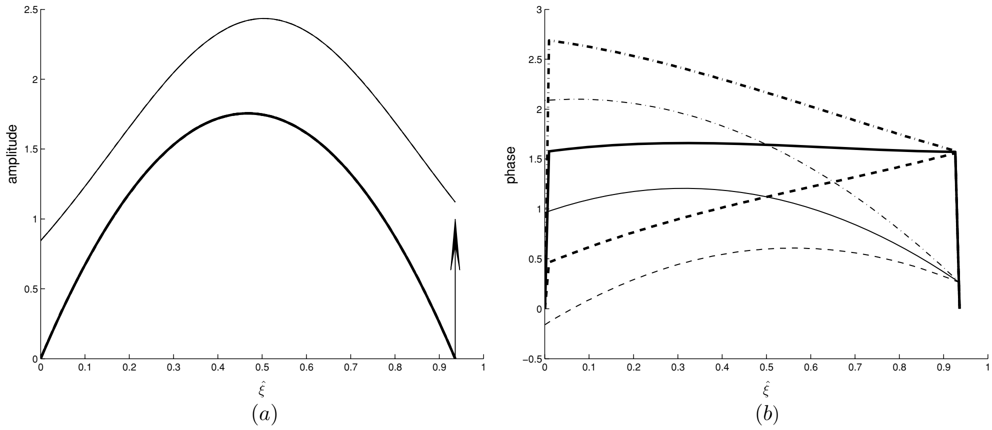

Figure 1. Green's function plots for a 19-period undulator. Exact formula (bold thickness), asymptotic approximation (normal thickness). $\hat{s} = 0$ (solid), $\hat{s} = 15$ (dashed), $\hat{s} = -15$ (dash-dotted). The Green's function amplitudes for different $\hat{s}$ coordinates are superposed. The phase is expressed in radians. (*a*) Amplitude comparison. (*b*) Phase comparison.

At first, we consider a short 19 periods undulator, such as that employed in the FERMI@Elettra project [8]. The normalized length of the undulator is $\hat{z} = 1.87$ and the linear chirp parameter is set to $\hat{\alpha} = 0.04$. Figure 1 shows the Green's functions obtained at the beam positions: $\hat{s}_0 = 0$, $\hat{s}_1 = 15$ and $\hat{s}_2 = -15$. The system is almost in the lethargy regime, and it can be noted that the approximated and exact expressions of the Green's functions are different in both amplitude and phase. Furthermore, the exact formula in (16) shows the presence of a Dirac delta pulse at $\hat{\xi} = \hat{z}/2$, which represents the laser seed moving at the velocity of light throughout the undulator. This allows a correct characterization of the group velocity of the FEL pulse in the letargy regime, as shown in figure 1 in [9].

As a second test case, we consider a 190-period undulator. The normalized length of the undulator is $\hat{z} = 18.7$, and the linear chirp parameter is set to $\hat{\alpha} = 0.01$. In this case, the system is not in the lethargy regime, and the asymptotic and the exact representations of the Green's function are in very good agreement. In particular, as shown in figure 2, the phase plots of the Green's function differ only for $\hat{\xi}$ coordinates where the gain is low. This result allows us to use also the asymptotic expression of the Green's function in the exponential gain regime, obtaining good estimates of both amplitude and phase of the FEL pulse.

In figure 3, we show the evolution of the centrovelocity of the FEL pulse, compared to the bunch bulk velocity v_b, the velocity of light and the theoretical exponential growth envelope velocity v_{th}. The centrovelocity is computed numerically using the relation

$$v_c(z) = \frac{z}{\langle t \rangle} = \frac{z \int_{-\infty}^{+\infty} E(t,z) E^*(t,z)\, \mathrm{d}t}{\int_{-\infty}^{+\infty} t E(t,z) E^*(t,z)\, \mathrm{d}t}. \tag{17}$$

Figure 3 shows the relative difference between the centrovelocity v_c and the velocity of light in the vacuum c, for the two different regimes of lethargy and exponential growth. Note, in particular, that in the lethargy regime v_c is close to the velocity of light, while in the exponential growth regime it approaches $v_{th} = v_b + (c - v_b)/3$, where $v_b = \frac{\omega_0}{k_0 + k_w}$.

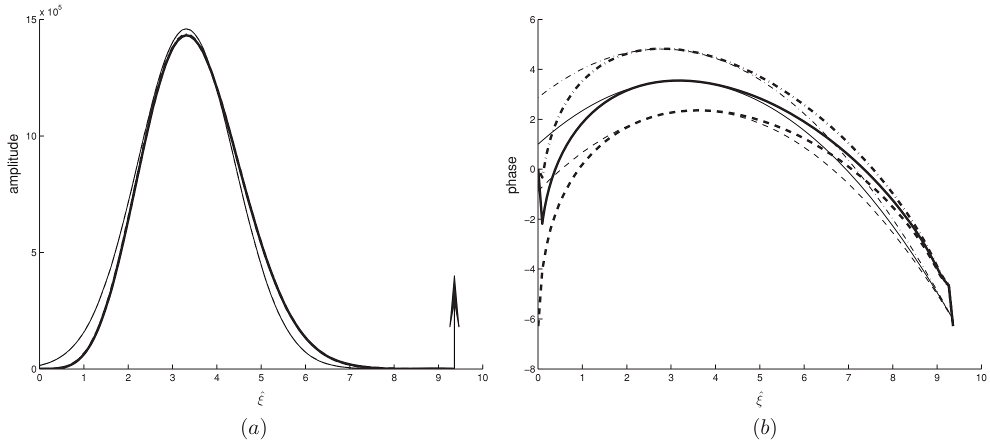

Figure 2. Green's function plots for a 190-period undulator. Exact formula (bold thickness), asymptotic approximation (normal thickness). $\hat{s} = 0$ (solid), $\hat{s} = 15$ (dashed), $\hat{s} = -15$ (dash-dotted). The Green's function amplitudes for different $\hat{s}$ coordinates are superposed. The phase is expressed in radians. (*a*) Amplitude comparison. (*b*) Phase comparison.

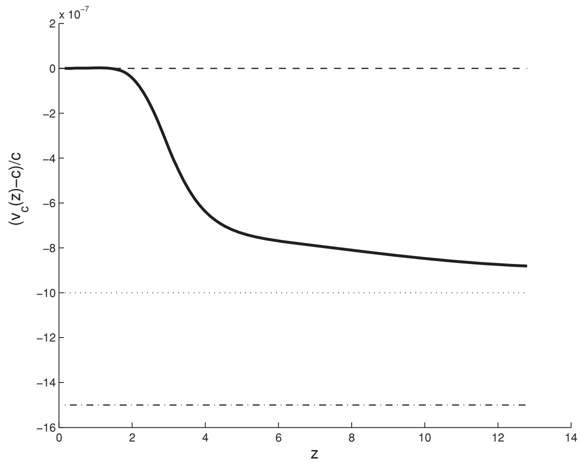

Figure 3. $(v_c - c)/c$ as a function of z. The solid thick line represents the centrovelocity of the FEL pulse. For comparison, the velocity of light is dashed, the theoretical velocity in exponential growth v_{th} is dotted and the bunch velocity v_b is dash-dotted.

4. Conclusion

An exact expression for the FEL Green's function of a seeded process involving an electron beam with a linear energy chirp has been derived by solving the coupled Vlasov–Maxwell equations. The new formula allows an accurate analysis of the evolution of the FEL pulse, in the presence of a chirped electron beam, in the lethargy start-up phase and in the exponential growth regime. The formula involves a Dirac delta pulse at $\hat{\xi} = \hat{z}/2$ which represents a seed advancing in the undulator at the velocity of light that allows to evaluate correctly the centrovelocity of the FEL pulse, particularly in the lethargy regime. The new formula also gives the correct Green's function for a wider range of values of the $\hat{\alpha}$ parameter, compared to the formula derived in [4]. This allows a correct evaluation of the FEL evolution in both amplitude and phase for electron beams having a larger energy chirp.

Appendix. An exact expression of the inverse Laplace transform representing the Green's function

We want to calculate the Green's function $I = \mathcal{G}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ given by (15). To this aim, let us introduce the auxiliary integral

$$I_{\mathrm{aux}} = \frac{1}{\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{e}^{\hat{p}(\hat{z}-2\hat{\xi})} \left(\mathrm{e}^{\frac{2\mathrm{i}\hat{\xi}}{(\hat{p}+\mathrm{i}\hat{\alpha}\hat{s})(\hat{p}+\mathrm{i}\hat{\alpha}(\hat{s}-\hat{\xi}))}} - 1 \right) \mathrm{d}\hat{p}, \tag{A.1}$$

which is related to I by the equation

$$I = I_{\mathrm{aux}} + 2\delta(\hat{z} - 2\hat{\xi}) \tag{A.2}$$

First of all, we observe that the integrand function in (A.1) has two singularities in $\hat{p}_1 = -\mathrm{i}\hat{\alpha}\hat{s}$ and $\hat{p}_2 = \mathrm{i}\hat{\alpha}(\hat{\xi} - \hat{s})$.

Furthermore, by the residual theorem and the Jordan's lemma, it results $I_{\mathrm{aux}} = 0$ for $\hat{z} - 2\hat{\xi} < 0$, while for $\hat{z} - 2\hat{\xi} > 0$, the quantity $I_{\mathrm{aux}}/2$ is equal to the sum of the residuals of the integrand function in the singularities $\hat{p}_1$ and $\hat{p}_2$. On the other hand, such residuals coincide with those of the integrand function in (15). Therefore, taking $\sigma > 0$, we can write

$$I_{\mathrm{aux}} = \begin{cases} 2 \sum_{j=1}^{2} \mathrm{Res}\left[\mathrm{e}^{\hat{p}(\hat{z}-2\hat{\xi}) + \frac{2\mathrm{i}\hat{\xi}}{(\hat{p}+\mathrm{i}\hat{\alpha}\hat{s})(\hat{p}+\mathrm{i}\hat{\alpha}(\hat{s}-\hat{\xi}))}}, \hat{p} = \hat{p}_j\right] & \text{for} \quad \hat{\xi} < \hat{z}/2 \\ 0 & \text{for} \quad \hat{\xi} > \hat{z}/2. \end{cases} \tag{A.3}$$

Therefore, for $\hat{\xi} < \hat{z}/2$, it results

$$I_{\mathrm{aux}} = 2\left(a_{-1}|_{\hat{p}_1} + a_{-1}|_{\hat{p}_2} \right), \tag{A.4}$$

where $a_{-1}|_{\hat{p}_j}$ is the multiplicative coefficient of the term $(\hat{p} - \hat{p}_j)^{-1}$ in the Laurent series expansion of the integrand function in a neighborhood of the singularity $\hat{p}_j$.

In order to find the coefficients $a_{-1}|_{\hat{p}_1}$ and $a_{-1}|_{\hat{p}_2}$, the integrand function in (15) can be expressed as

$$\begin{aligned} \mathrm{e}^{\hat{p}(\hat{z}-2\hat{\xi}) + \frac{2\mathrm{i}\hat{\xi}}{(\hat{p}+\mathrm{i}\hat{\alpha}\hat{s})(\hat{p}+\mathrm{i}\hat{\alpha}(\hat{s}-\hat{\xi}))}} &= \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\hat{p}(\hat{z} - 2\hat{\xi}) + \frac{2\mathrm{i}\hat{\xi}}{(\hat{p} + \mathrm{i}\hat{\alpha}\hat{s})(\hat{p} + \mathrm{i}\hat{\alpha}(\hat{s} - \hat{\xi}))} \right)^n \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \hat{p}^k (\hat{z} - 2\hat{\xi})^k \frac{(2\mathrm{i}\hat{\xi})^{n-k}}{(\hat{p} - \hat{p}_1)^{n-k} (\hat{p} - \hat{p}_2)^{n-k}}, \end{aligned} \tag{A.5}$$

where the known expansion formula of the power of a binomial term has been used. To calculate $a_{-1}|_{\hat{p}_1}$, in (A.5), we introduce the variable change $\hat{p} = x + \hat{p}_1$, and to calculate $a_{-1}|_{\hat{p}_2}$ we set $\hat{p} = x + \hat{p}_2$, obtaining respectively

$$\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} (x + \hat{p}_1)^k (\hat{z} - 2\hat{\xi})^k \frac{(2\mathrm{i}\hat{\xi})^{n-k}}{x^{n-k} (x + (\hat{p}_1 - \hat{p}_2))^{n-k}} \tag{A.6}$$

$$\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} (x + \hat{p}_2)^k (\hat{z} - 2\hat{\xi})^k \frac{(2\mathrm{i}\hat{\xi})^{n-k}}{x^{n-k} (x + (\hat{p}_2 - \hat{p}_1))^{n-k}}. \tag{A.7}$$

Equations (A.6) and (A.7) can be written in the form

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{(\hat{z}-2\hat{\xi})^k(2\mathrm{i}\hat{\xi})^{n-k}}{(n-k)!k!}\frac{1}{x^{n-k}}\frac{(x+\hat{p}_j)^k}{(x+\hat{p}_j-\hat{p}_{3-j})^{n-k}} \tag{A.8}$$

setting $j=1$ and $j=2$, respectively. In order to calculate the coefficient for x^{-1}, we consider the Taylor series expansion of the function $\frac{(x+\hat{p}_j)^k}{(x+\hat{p}_j-\hat{p}_{3-j})^{n-k}}$ in (A.8), in the neighborhood of $x=0$:

$$\frac{(x+\hat{p}_j)^k}{(x+\hat{p}_j-\hat{p}_{3-j})^{n-k}}=\sum_{w=0}^{+\infty}\frac{1}{w!}\frac{\mathrm{d}^w}{\mathrm{d}x^w}\frac{(x+\hat{p}_j)^k}{(x+\hat{p}_j-\hat{p}_{3-j})^{n-k}}\bigg|_{x=0}x^w, \tag{A.9}$$

and we calculate the multiplicative coefficient for $x^w=x^{n-k-1}$. Note that, since $w\geqslant 0$, we have $k\leqslant n-1$. After some algebraic manipulations, the coefficient for x^w results to be given by

$$\begin{aligned}H(n,k,\hat{p}_j,\hat{p}_{3-j})&=\frac{1}{(n-k-1)!}\frac{\mathrm{d}^{(n-k-1)}}{\mathrm{d}x^{(n-k-1)}}\frac{(x+\hat{p}_j)^k}{(x+\hat{p}_j-\hat{p}_{3-j})^{n-k}}\bigg|_{x=0}\\&=\sum_{g=0}^{\min[n-k-1,k]}\frac{(-1)^{n-k-1-g}}{g!(n-k-1-g)!}\frac{k!(2n-2k-g-2)!}{(n-k-1)!(k-g)!}\\&\quad\times\hat{p}_j^{k-g}(\hat{p}_j-\hat{p}_{3-j})^{-2n+2k+1+g}.\end{aligned} \tag{A.10}$$

The coefficient $a_{-1}|_{\hat{p}_j}$ is then given by

$$a_{-1}|_{\hat{p}_j}=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!k!}(2\mathrm{i}\hat{\xi})^{n-k}(\hat{z}-2\hat{\xi})^kH(n,k,\hat{p}_j,\hat{p}_{3-j}). \tag{A.11}$$

Substituting (A.11) into (A.4) yields

$$I_{\mathrm{aux}}=2\sum_{n=0}^{+\infty}\sum_{k=0}^{n-1}\frac{(2\mathrm{i}\hat{\xi})^{n-k}(\hat{z}-2\hat{\xi})^k}{(n-k)!k!}\left(H(n,k,\hat{p}_1,\hat{p}_2)+H(n,k,\hat{p}_2,\hat{p}_1)\right). \tag{A.12}$$

To simplify (A.12), we observe that both $\hat{p}_1$ and $\hat{p}_2$ are proportional to $-\mathrm{i}\hat{\alpha}$. Setting $\tilde{\hat{p}}_1=\frac{\hat{p}_1}{-\mathrm{i}\hat{\alpha}}=\hat{s}$ and $\tilde{\hat{p}}_2=\frac{\hat{p}_2}{-\mathrm{i}\hat{\alpha}}=\hat{s}-\hat{\xi}$, we rewrite I_{aux} as

$$I_{\mathrm{aux}}=2\sum_{n=0}^{+\infty}\sum_{k=0}^{n-1}\frac{(2\mathrm{i}\hat{\xi})^{n-k}(\hat{z}-2\hat{\xi})^k}{(n-k)!k!}(-\mathrm{i}\hat{\alpha})^{3k-2n+1}(H(n,k,\tilde{\hat{p}}_1,\tilde{\hat{p}}_2)+H(n,k,\tilde{\hat{p}}_2,\tilde{\hat{p}}_1)), \tag{A.13}$$

which using (A.10) becomes

$$\begin{aligned}I_{\mathrm{aux}}&=\sum_{n=0}^{+\infty}\sum_{k=0}^{n-1}\frac{(\hat{z}-2\hat{\xi})^k2^{n-k+1}\mathrm{i}^{n-2k+1}\hat{\alpha}^{3k-2n+1}\hat{\xi}^{n-k}}{(n-k)!(n-k-1)!}\sum_{g=0}^{\min[n-k-1,k]}\frac{(-1)^g(2n-2k-g-2)!}{g!(n-k-1-g)!(k-g)!}\\&\quad\times\left(\tilde{\hat{p}}_1^{k-g}(\tilde{\hat{p}}_1-\tilde{\hat{p}}_2)^{-2n+2k+1+g}+\tilde{\hat{p}}_2^{k-g}(\tilde{\hat{p}}_2-\tilde{\hat{p}}_1)^{-2n+2k+1+g}\right).\end{aligned} \tag{A.14}$$

Since $\tilde{\hat{p}}_1=\hat{s}$ and $\tilde{\hat{p}}_2=\hat{s}-\hat{\xi}$, (A.14) can be written as

$$\begin{aligned}I_{\mathrm{aux}}&=\sum_{n=0}^{+\infty}\sum_{k=0}^{n-1}\frac{(\hat{z}-2\hat{\xi})^k2^{n-k+1}\mathrm{i}^{n-2k+1}\hat{\alpha}^{3k-2n+1}\hat{\xi}^{k+1-n}}{(n-k)!(n-k-1)!}\\&\quad\times\sum_{g=0}^{\min[n-k-1,k]}\frac{\hat{\xi}^g((-1)^g\hat{s}^{k-g}-(\hat{s}-\xi)^{k-g})(2n-2k-g-2)!}{g!(n-k-1-g)!(k-g)!}\end{aligned} \tag{A.15}$$

Introducing the index changes

$$n = 3h + t + 2; \qquad k = 2h + t + 1 \tag{A.16}$$

and recalling that $0 \leqslant k \leqslant n - 1$, we obtain the inequalities

$$h \geqslant 0; \qquad t \geqslant -2h - 1. \tag{A.17}$$

Thus, we can write

$$I_{\mathrm{aux}} = \sum_{h=0}^{+\infty} \sum_{t=-2h-1}^{+\infty} \frac{(\hat{z} - 2\hat{\xi})^{1+2h+t} 2^{2+h} \mathrm{i}^{-h-t-1} \hat{\alpha}^t \hat{\xi}^{-h}}{h!(h+1)!} \times \sum_{g=0}^{\min[h,1+2h+t]} \frac{(2h-g)! \hat{\xi}^g ((-1)^g \hat{s}^{1+2h+t-g} - (\hat{s} - \hat{\xi})^{1+2h+t-g})}{g!(h-g)!(2h+t+1-g)!}. \tag{A.18}$$

We now show that the multiplicative coefficients of the terms $\hat{\alpha}^r$ with $r < 0$ are zero. To this aim, let us indicate by $f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ the integrand in equation (15), that is

$$f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \mathrm{e}^{\hat{p}(\hat{z} - 2\hat{\xi}) + \frac{2\mathrm{i}\hat{\xi}}{(\hat{p} + \mathrm{i}\hat{\alpha}\hat{s})(\hat{p} + \mathrm{i}\hat{\alpha}(\hat{s} - \hat{\xi}))}}. \tag{A.19}$$

Assuming that $\hat{s}$, $\hat{\xi}$ and $\hat{z}$ are assigned, the integration problem posed above can be reduced to that of calculating, for any real $\hat{\alpha}$, the sum of the residual values of $f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ in the two singularities, given by

$$\mathrm{Res}(\hat{p}_1) + \mathrm{Res}(\hat{p}_2) = \frac{1}{2\pi \mathrm{i}} \oint_C f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})\, \mathrm{d}\hat{p}, \tag{A.20}$$

where the integration line C includes, as inner points, the singularities $\hat{p}_1$ and $\hat{p}_2$ as well as the origin of the complex plane $\hat{p}$. Therefore, there exists a neighborhood U of $\hat{\alpha} = 0$ where, for any $\hat{p} \in C$, the function $f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ is bounded and has continuous derivatives of any order with respect to $\hat{\alpha}$. Therefore, it can be expanded in Taylor series

$$f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n f}{\partial \hat{\alpha}^n} \right)_{\hat{\alpha}=0} \hat{\alpha}^n, \tag{A.21}$$

where all of the derivatives depend on $\hat{p}$ ($\hat{s}$, $\hat{\xi}$ and $\hat{z}$ are assigned). Substituting (A.21) into (A.20) and integrating yields

$$\mathrm{Res}(\hat{p}_1) + \mathrm{Res}(\hat{p}_2) = \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\oint_C \left(\frac{\partial^n f}{\partial \hat{\alpha}^n} \right)_{\hat{\alpha}=0} \mathrm{d}\hat{p} \right] \hat{\alpha}^n. \tag{A.22}$$

The right-hand side of equation (A.22) does not involve terms $\hat{\alpha}^n$ with $n < 0$. Therefore, recalling (A.4), the same holds for the series expression in equation (A.18). Taking this into account, (A.18) can be written as

$$I_{\mathrm{aux}} = \sum_{h=0}^{+\infty} \sum_{t=0}^{+\infty} \frac{(\hat{z} - 2\hat{\xi})^{1+2h+t} 2^{2+h} \mathrm{i}^{-h-t-1} \hat{\alpha}^t \hat{\xi}^{-h}}{h!(h+1)!} \times \sum_{g=0}^{h} \frac{(2h-g)! \hat{\xi}^g ((-1)^g \hat{s}^{1+2h+t-g} - (\hat{s} - \hat{\xi})^{1+2h+t-g})}{g!(h-g)!(2h+t+1-g)!}. \tag{A.23}$$

The latter expression can be further simplified into the following:

$$I_{\mathrm{aux}}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \sum_{t=0}^{+\infty} \hat{\alpha}^t \sum_{h=0}^{+\infty} \frac{\mathrm{i}^{1+h+t} 2^{2+h} (\hat{z} - 2\hat{\xi})^{1+2h+t} \sum_{w=0}^{t} \frac{(-1)^w \hat{s}^w \hat{\xi}^{1+h+t-w} (h+t-w)!}{w!(t-w)!(1+2h+t-w)!}}{h!(1+h)!}.$$

The Green's function is finally expressed as

$$\mathcal{G}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \delta(\hat{\xi} - \hat{z}/2) + I_{\mathrm{aux}}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}), \tag{A.24}$$

which corresponds to equation (16).

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